

Toyama Math. J.
Vol. 37(2015), 55–72

Module structure on $\mathcal{U}(H)$ for basic Lie superalgebras

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Dedicated to Prof. Jun Morita on the occasion of his 60th birthday

Abstract. We study the category \mathcal{M} consisting of modules whose restriction to $\mathcal{U}(H)$ is free of rank 1 for the basic Lie superalgebras. We show that \mathcal{M} is not empty only for the Lie superalgebra $B(0, n) = \mathfrak{osp}(1|2n)$. We classify the isomorphism classes of objects in \mathcal{M} for $\mathfrak{osp}(1|2n)$ and determine their irreducibility. This leads to a lot of new modules over $\mathfrak{osp}(1|2n)$.

1. Introduction

Classification of simple modules is important in understanding the representation theory of an algebra. Since the 1970's, the representation theory for the basic Lie superalgebras has been extensively studied by I.N. Bernstein, I. Dimitrov, V. Kac, D.A. Leites, O. Mathieu, I. Mussion, I. Penkov, V. Serganova, J. Van der Jeugt, W. Wang and others (see [1]–[8], [10]–[14], [17], [20] and references therein).

One of the most classical families of modules is the family of so-called weight modules. They are the modules on which the Cartan subalgebra acts diagonalizably. Recently, J. Nilson, Tan and Zhao studied the modules defined by the “opposite condition” for the Lie algebras \mathfrak{sl}_{n+1} (and the Witt algebras W_n, W_n^+), namely, the \mathfrak{sl}_{n+1} -modules which are free of rank 1 when restricted to the Cartan algebra (see [15], [18] and [19]). Such modules for

2000 *Mathematics Subject Classification.* 17B10, 17B65, 17B68.

Key words and phrases. basic Lie superalgebras, Cartan subalgebras, simple modules, free modules.

finite dimensional Lie algebras were studied by J. Nilson (see [16]). In [6], the authors studied such modules for all Kac-Moody algebras. Chen and Guo studied such modules for Heisenberg algebra and the Lie algebra $W(2, 2)$ in [3]. In the present paper, we will study such modules for the basic Lie superalgebras, especially for the Lie superalgebra $\mathfrak{osp}(1|2n)$, which plays an exceptional roles in the theory of Lie superalgebras (see [12]). The Lie superalgebra $\mathfrak{osp}(1|2n)$ has some extraordinary properties among the basic Lie superalgebra. For example, it is the only simple Lie superalgebra whose category of finite dimensional representations is semisimple. In this paper, we will show that it is the only basic Lie superalgebra that admits modules which are free of rank 1 when restricted to its Cartan subalgebra.

The paper is organized as follow. In section 2, we give some notations and basic definitions. We also prove some useful results and show that $\mathfrak{osp}(1|2n)$ is the only basic Lie superalgebra that admits such modules. In section 3, we determine the module structure for $\mathfrak{osp}(1|2n)$ and classify their isomorphism classes.

In this paper, we always denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ and \mathbb{C}^* the set of integers, nonnegative integers, positive integers, complex numbers and nonzero complex numbers respectively.

2. Preliminaries

First of all, let us recall some definitions and notations which will be used in our discussion.

2.1. Basic Lie superalgebras

A Lie superalgebra G is a \mathbb{Z}_2 -graded vector space $G = G_{\bar{0}} \oplus G_{\bar{1}}$ endowed with a bilinear map $[\cdot, \cdot] : G \times G \rightarrow G, (X, Y) \mapsto [X, Y]$, called the Lie superbracket, which is homogeneous of degree 0, graded skew-symmetric and satisfies the super Jacobian identity. The classification of finite dimensional Lie superalgebras was complete in the late 70s (see [12]). In particular, we have the following well-known

Theorem 2.1. *The basic Lie superalgebras are the simple contragredient*

Lie superalgebras:

$$\begin{aligned}
A(m, n) &= \mathfrak{sl}(m+1|n+1) \text{ with } m > n \geq 0; \\
A(n, n) &= \mathfrak{psl}(n+1|n+1) \text{ with } n \geq 1; \\
B(m, n) &= \mathfrak{osp}(2m+1|2n) \text{ with } m \geq 0, n > 0; \\
C(n) &= \mathfrak{osp}(2|2n) \text{ with } n \geq 2; \\
D(m, n) &= \mathfrak{osp}(2m|2n) \text{ with } m \geq 2, n \geq 1; \\
D(2, 1, \alpha) &\text{ with } \alpha \in \mathbb{C} \setminus \{0, -1\}; \\
F(4), G(3).
\end{aligned}$$

Let G be a basic Lie superalgebra and $T = \{1, 2, \dots, s\}$. From [9], we know that there exists a set of generators $\{e_i, f_i, H_i | i \in T\}$ of G , a subset τ of T and a matrix $A = (a_{ij})$ such that

- (1) $H_i \in G_0$;
- (2) $e_i, f_i \in G_0$ if $i \notin \tau$ and $e_i, f_i \in G_{\bar{1}}$, if $i \in \tau$;
- (3) $[H_i, H_j] = 0$;
- (4) $[e_i, f_j] = \delta_{ij} H_i$;
- (5) $[H_j, e_i] = a_{ji} e_i, [H_j, f_i] = -a_{ji} f_i$;
- (6) $[e_i, e_i] = [f_i, f_i] = 0$, if $a_{ii} = 0$.

The matrix A is called the Cartan matrix of G and the abelian subalgebra $H = \text{span}_{\mathbb{C}}\{H_i | i \in T\} \subseteq G_0$ is the Cartan subalgebra of G . Indeed, let

$$A_i = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{i \times i}, C_i = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}_{i \times i}$$

$$D_i = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}_{i \times i}, v_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}_{n \times 1}, v_2 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1}$$

then Cartan matrices for basic Lie superalgebras are as below:

Lie superalgebra	s	τ	Cartan matrix
$A(m, n)$	$m + n + 1$	$\{m + 1\}$	$\begin{pmatrix} A_m & v_1 & O \\ v_1^t & 0 & -v_2^t \\ O & v_2 & A_n \end{pmatrix}$
$B(0, n)$	n	$\{n\}$	B
$B(m, n)(m > 0)$	$n + m$	$\{n\}$	$\begin{pmatrix} A_{n-1} & v_1 & O \\ v_1^t & 0 & -v_2^t \\ O & v_2 & C_n \end{pmatrix}$
$C(n)(n > 2)$	n	$\{1\}$	$\begin{pmatrix} 0 & -v_2^t \\ v_2 & C_n^t \end{pmatrix}$
$D(m, n)(m > 1)$	$n + m$	$\{n\}$	$\begin{pmatrix} A_{n-1} & v_1 & O \\ v_1^t & 0 & -v_2^t \\ O & v_2 & D_m \end{pmatrix}$
$F(4)$	4	$\{1\}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$
$G(3)$	3	$\{1\}$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$
$D(2, 1, \alpha)(\alpha \in \mathbb{C} \setminus \{0, -1\})$	3	$\{1\}$	$\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$

It is easy to see that the Cartan matrix for a basic Lie superalgebra which is not of type $A(n, n)$ is invertible. For a basic Lie superalgebra which is

not of type $A(n, n)$, let

$$\begin{pmatrix} h_1 \\ \vdots \\ h_s \end{pmatrix} = A^{-1} \begin{pmatrix} H_1 \\ \vdots \\ H_s \end{pmatrix}$$

Then $\{h_i | i \in T\}$ is a basis for H and

$$[h_i, e_j] = \delta_{ij} e_j, [h_i, f_j] = -\delta_{ij} f_j, [e_i, f_j] = \delta_{ij} H_i \in H, \forall i, j \in T. \quad (2.1)$$

Also, we can identify $\mathcal{U}(H) \cong \mathbb{C}[h_1, \dots, h_s] \cong \mathbb{C}[H_1, \dots, H_s]$.

2.2. Categories

Let G be a basic Lie superalgebra with Cartan subalgebra H . Define \mathcal{M} to be the subcategory of $\mathcal{U}(G)\text{-Mod}$ consisting of objects whose restriction to $\mathcal{U}(H)$ are free of rank 1:

$$\mathcal{M} := \{M \in \mathcal{U}(G)\text{-Mod} \mid \text{Res}_{\mathcal{U}(H)}^{\mathcal{U}(G)} M \cong_{\mathcal{U}(H)} \mathcal{U}(H)\}$$

where we denote by $\mathcal{U}(\mathfrak{g})\text{-Mod}$ the category of $\mathcal{U}(\mathfrak{g})$ -modules for any algebra \mathfrak{g} .

Let $\{h_i, e_i, f_i | i \in T\}$ be a set of generators of G . To describe our results, we need some notations on functors. Each automorphism $\psi \in \text{Aut}(\mathcal{U}(G))$ induces a functor

$$F_\psi : \mathcal{U}(G)\text{-Mod} \longrightarrow \mathcal{U}(G)\text{-Mod}$$

which maps each module to itself but with a new action \bullet defined by: $x \bullet v := \psi(x) \cdot v$. The functor F_ψ maps morphisms to themselves. Let $(\mathbb{C}^*)^s$ be the direct product of s copies of the multiplicative group \mathbb{C}^* . For any $a = (a_1, \dots, a_s) \in (\mathbb{C}^*)^s$, let $\psi_a \in \text{Aut}(\mathcal{U}(G))$ be the automorphism induced by

$$\psi_a : e_i \mapsto a_i e_i; f_i \mapsto \frac{1}{a_i} f_i; h_i \mapsto h_i$$

and denoted by F_a the functor F_{ψ_a} .

Here we list some properties of F_a .

Lemma 2.2. *The functors F_a has the following properties:*

- (1) $F_{(1, \dots, 1)} = \text{Id}_{\mathcal{U}(G)\text{-Mod}}$;
- (2) $F_a \circ F_b = F_{ab}$;
- (3) $F_a^{-1} = F_{a^{-1}}$;
- (4) F_a is an auto-equivalence;
- (5) $F_a(\mathcal{M}) = \mathcal{M}$;
- (6) For any $M \in \mathcal{M}$, $F_a(M) \cong M$ if and only if $a = (1, \dots, 1)$.

Proof. The proof for claim (1)-(4) is similar to that of Lemma 7 in [15]. We now prove claim (5) and (6). Let $M \in \mathcal{M}$.

(5) We first note that $F_a(M)$ is still equal to M as a set, and the action of H is the same since φ_a fixes $\mathcal{U}(H)$ so $F_a(M)$ is still free of rank 1 over $\mathcal{U}(H)$.

(6) Suppose $\Phi : F_a(M) \rightarrow M$ is an isomorphism. Since $\Phi(g) = g \bullet \Phi(1) = g \cdot \Phi(1)$, Φ is determined by $\Phi(1)$ and the same is true for Φ^{-1} . From $1 = \Phi^{-1}(\Phi(1)) = \Phi^{-1}(1)\Phi(1)$, we see that $\Phi(1) = c \in \mathbb{C}^*$ and thus $\Phi(g) = cg, \forall g \in M$. However, for any $i \in T$

$$\begin{aligned} c(e_i \cdot 1) &= e_i \cdot \Phi(1) = \Phi(e_i \bullet 1) = \Phi(a_i e_i \cdot 1) = ca_i e_i \cdot 1, \\ c(f_i \cdot 1) &= f_i \cdot \Phi(1) = \Phi(f_i \bullet 1) = \Phi(a_i^{-1} f_i \cdot 1) = ca_i^{-1} f_i \cdot 1, \end{aligned}$$

where $e_i \cdot 1, f_i \cdot 1$ can not be all zero, as otherwise $H_i \cdot 1 = [e_i, f_i] \cdot 1 = 0$ which is impossible. Hence, we have $a_i = 1, \forall i \in T$. \square

Let $M \in \mathcal{M}$. Since M is free of rank 1 as $\mathcal{U}(H)$ -module, we may assume that $M = \mathcal{U}(H)$ as the natural $\mathcal{U}(G)$ -module. Thus, to classify the isomorphism classes of objects in \mathcal{M} , we need only consider all possible extension of the natural left $\mathcal{U}(H)$ -action on $\mathcal{U}(H)$ to $\mathcal{U}(G)$.

2.3. Actions of generators

In this subsection, we assume that G is not of type $A(n, n)$. To describe the action of $\mathcal{U}(G)$ on $M = \mathcal{U}(H)$, we need to introduce some notations for

our polynomial rings. Define

$$\begin{aligned}\mathcal{P} &:= \mathbb{C}[h_1, \dots, h_s], \\ \mathcal{P}_i &:= \mathbb{C}[h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_s], \forall i \in T\end{aligned}$$

Note that $\mathcal{U}(H) = \mathcal{P}$.

We define for each $i \in T$ an algebra automorphism

$$\sigma_i : \mathcal{U}(H) \rightarrow \mathcal{U}(H)$$

by $\sigma_i(h_k) := h_k - \delta_{ik}$. Explicitly, we have

$$\sigma_i(g) = g(h_1, \dots, h_i - 1, \dots, h_s)$$

Remark 2.3. Obviously, we have $\sigma_i \sigma_j = \sigma_j \sigma_i$ for any i, j .

In [15], the author proved that

Lemma 2.4. Let $g \in \mathcal{P}$ be a nonzero polynomial, then

$$\deg_i(\sigma_i(g) - g) = \deg_i g - 1$$

where $\deg_i 0$ is defined to be -1 .

Proposition 2.5. Let $M \in \mathcal{M}$. Then identifying M with \mathcal{P} as vector spaces, the action of G on M is completely determined by the action of the generators $\{h_i, e_i, f_i | i \in T\}$ on $1 \in M$. Explicitly, for $g \in \mathcal{P}, k \in T$, we have

$$\begin{aligned}h_k \cdot g &= h_k g, \\ e_k \cdot g &= p_k \sigma_k(g), \\ f_k \cdot g &= q_k \sigma_k^{-1}(g)\end{aligned}$$

where $p_k := e_k \cdot 1$ and $q_k := f_k \cdot 1$.

Proof. We know that $M = \mathcal{P}$ as the natural H -module. That is, the action of H on M can be written explicitly as $h_k \cdot g = h_k g$ for all $g \in M, k \in T$.

Now for each $k \in T$, define

$$p_k := e_k \cdot 1, q_k := f_k \cdot 1.$$

Since $\delta_{ki}(e_k \cdot g) = [h_i, e_k] \cdot g = h_i(e_k \cdot g) - e_k \cdot (h_i g)$, we see that

$$e_k \cdot (h_i g) = (h_i - \delta_{ki})(e_k \cdot g)$$

which shows that the action of e_k can be determined inductively from its action on 1. Explicitly, we get $e_k \cdot g = p_k \sigma_k(g)$. Similarly, we can get $f_k \cdot g = q_k \sigma_k^{-1}(g)$.

Since $\{h_i, e_i, f_i | i \in T\}$ generates G , the action of G on M is completely determined by the $(2s)$ -tuple $(p_1, \dots, p_s, q_1, \dots, q_s) \in \mathcal{P}^{2s}$ as stated in the proposition. \square

Note that not every tuple $(p_1, \dots, p_s, q_1, \dots, q_s)$ determines a G -module. We now turn to the converse problem: determine which choices of $(p_1, \dots, p_s, q_1, \dots, q_s) \in \mathcal{P}^{2s}$ give rise to a module structure on \mathcal{P} by the definition of the action in Proposition 2.5.

2.4. Results for basic Lie superalgebra not of type $B(0, n)$

If G is a basic Lie superalgebra which is not of type $B(0, n)$ or $A(n, n)$, then from the Cartan matrix, we know that G has a set of generators satisfying (2.1) and

$$\{e_k, e_k\} = \{f_k, f_k\} = 0$$

for some $k \in \tau$. Let $M \in \mathcal{M}$ with module actions:

$$\begin{aligned} h_i \cdot g &= h_i g, \\ e_i \cdot g &= p_i \sigma_i(g), \\ f_i \cdot g &= q_i \sigma_i^{-1}(g) \end{aligned}$$

where $i \in T, g \in \mathcal{P}$. Then, we have

$$p_k \sigma_k(p_k) = q_k \sigma_k^{-1}(q_k) = 0$$

which implies that

$$h_k g = 0, \forall g \in \mathcal{P}.$$

This is impossible. Thus, we have

Proposition 2.6. *Let G be a basic Lie superalgebra and suppose that $G \neq \mathfrak{osp}(1|2n)$. Then*

$$\mathcal{M} = \emptyset.$$

Proof. To complete the proof, we need to show the statement for the basic Lie superalgebra $\mathfrak{psl}(n, n)$. Similar to Proposition 2.5, we have

$$\begin{aligned} & e_{n+1} \cdot g(H_1, \dots, H_n, H_{n+1}, H_{n+2}, \dots, H_{2n+1}) \\ &= pg(H_1, \dots, H_n + 1, H_{n+1}, H_{n+2} - 1, \dots, H_{2n+1}), \\ & f_{n+1} \cdot g(H_1, \dots, H_n, H_{n+1}, H_{n+2}, \dots, H_{2n+1}) \\ &= qg(H_1, \dots, H_n - 1, H_{n+1}, H_{n+2} + 1, \dots, H_{2n+1}), \end{aligned}$$

where $p = e_{n+1} \cdot 1, q = f_{n+1} \cdot 1$.

Following from

$$\{e_{n+1}, e_{n+1}\} = \{f_{n+1}, f_{n+1}\} = 0,$$

we have

$$\begin{aligned} & p(H_1, \dots, H_n, H_{n+1}, H_{n+2}, \dots, H_{2n+1}) \\ & \cdot p(H_1, \dots, H_n + 1, H_{n+1}, H_{n+2} - 1, \dots, H_{2n+1}) = 0, \\ & q(H_1, \dots, H_n, H_{n+1}, H_{n+2}, \dots, H_{2n+1}) \\ & \cdot q(H_1, \dots, H_n - 1, H_{n+1}, H_{n+2} + 1, \dots, H_{2n+1}) = 0. \end{aligned}$$

Hence,

$$p = q = 0.$$

Therefore,

$$H_{n+1} \cdot g = \{e_{n+1}, f_{n+1}\} \cdot g = 0.$$

This contradicts with the definition of \mathcal{M} . □

3. Results for the Lie superalgebra $B(0, n)$

For the rest of this paper, we always assume that G is the Lie superalgebra $B(0, n) = \mathfrak{osp}(1|2n)$. Let S be the set $\{0, 1, \dots, n, \bar{1}, \dots, \bar{n}\}$ with total order

$$0 < 1 < \dots < n < \bar{1} < \dots < \bar{n}.$$

Denote by $e_{IJ}(I, J \in S)$ the $(2n+1) \times (2n+1)$ matrix with zeros everywhere except a 1 on position (I, J) . Let $i = 1, \dots, n$. The basis of G is given by

$$\begin{aligned} E_{0i} &= E_{i0} = e_{i0} + e_{0\bar{i}} := E_i, \\ E_{0\bar{i}} &= E_{\bar{i}0} = e_{\bar{i}0} - e_{0i} := -F_i, \\ E_{ij} &= e_{i\bar{j}} + e_{j\bar{i}}, E_{\bar{i}\bar{j}} = -(e_{\bar{i}j} + e_{\bar{j}i}), \\ E_{i\bar{j}} &= E_{\bar{j}i} = e_{ij} - e_{j\bar{i}}, \end{aligned}$$

where $\{E_i, F_i | 1 \leq i \leq n\}$ are odd elements while $\{E_{IJ} | I, J \in S \setminus \{0\}\}$ are even ones. Moreover, $\{h_i := [E_i, F_i] = E_{\bar{i}\bar{i}} | 1 \leq i \leq n\}$ forms a basis for the Cartan subalgebra H and the even part $G_{\bar{0}} = \text{span}\{E_{IJ} | I, J \in S \setminus \{0\}\}$ is a Lie algebra of type C_n and $H \subseteq G_{\bar{0}}$. In particular, $\{E_{i, \bar{i}+1}, E_{i+1, \bar{i}}, E_{nn}, E_{\bar{n}\bar{n}} | 1 \leq i \leq n-1\}$ is a set of generators for $G_{\bar{0}}$. Thus, for any $M \in \mathcal{M}(G)$, we have $\text{Res}_{G_{\bar{0}}}^G M \in \mathcal{M}(G_{\bar{0}})$. The category \mathcal{M} for Lie algebras of type $C_n (n \geq 2)$ was determined by the authors in [6] and [16].

Theorem 3.1. *Let \mathfrak{g} be a complex Lie algebra of type C_n with basis $\{E_{IJ} | I, J \in S \setminus \{0\}\}$. Let $R \subseteq T = \{1, 2, \dots, n\}$. Let M_R be the set \mathcal{P} with the following \mathfrak{g} -action: for any $g \in M_R$, $1 \leq i \leq n-1$,*

$$\begin{aligned} h_k \cdot g &= h_k g, 1 \leq k \leq n, \\ E_{i, \bar{i}+1} \cdot g &= \begin{cases} (h_{i+1} + \frac{1}{2})\sigma_i \sigma_{i+1}^{-1}(g), & i, i+1 \in R, \\ \sigma_i \sigma_{i+1}^{-1}(g), & i \in R, i+1 \notin R, \\ (h_i - \frac{1}{2})(h_{i+1} + \frac{1}{2})\sigma_i \sigma_{i+1}^{-1}(g), & i \notin R, i+1 \in R, \\ (h_i - \frac{1}{2})\sigma_i \sigma_{i+1}^{-1}(g), & i, i+1 \notin R; \end{cases} \\ E_{i+1, \bar{i}} \cdot g &= \begin{cases} (h_i + \frac{1}{2})\sigma_i^{-1} \sigma_{i+1}(g), & i, i+1 \in R, \\ (h_i + \frac{1}{2})(h_{i+1} - \frac{1}{2})\sigma_i^{-1} \sigma_{i+1}(g), & i \in R, i+1 \notin R, \\ \sigma_i^{-1} \sigma_{i+1}(g), & i \notin R, i+1 \in R, \\ (h_{i+1} - \frac{1}{2})\sigma_i^{-1} \sigma_{i+1}(g), & i, i+1 \notin R; \end{cases} \quad (3.1) \\ E_{nn} \cdot g &= \begin{cases} \sigma_n^2(g), & n \in R, \\ (h_n - \frac{1}{2})(h_n - \frac{3}{2})\sigma_n^2(g), & n \notin R, \end{cases} \\ E_{\bar{n}\bar{n}} \cdot g &= \begin{cases} (h_n + \frac{3}{2})(h_n + \frac{1}{2})\sigma_n^{-2}(g), & n \in R, \\ \sigma_n^{-2}(g), & n \notin R. \end{cases} \end{aligned}$$

Then $M_R \in \mathcal{M}(\mathfrak{g})$ and

$$\mathcal{M}(\mathfrak{g}) = \{F_a(M_R) | a \in (\mathbb{C}^*)^n, R \subseteq T\}.$$

Moreover, every module in $\mathcal{M}(\mathfrak{g})$ is simple.

For Lie algebra of type C_1 , its category \mathcal{M} is as follow.

Theorem 3.2. Let $\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C})$ and $H = \frac{1}{2}h_1$. For any $S \subseteq \{1, 2\}$, $b \in \mathbb{C}$ and $a \in \mathbb{C}^*$ The set $M(a, b, S) = \mathbb{C}[H]$ becomes a \mathfrak{g} -module under the following actions $\forall g \in \mathbb{C}[H]$,

$$\begin{aligned} H \cdot g &= Hg; \\ E_{11} \cdot g &= \begin{cases} a(H+b)\sigma(g), & \text{if } 1, 2 \notin S \text{ or } 1, 2 \in S, \\ a\sigma(g), & \text{if } 1 \in S, 2 \notin S, \\ a(-H^2 + H + b)\sigma(g), & \text{if } 1 \notin S, 2 \in S; \end{cases} \\ E_{\bar{1}\bar{1}} \cdot g &= \begin{cases} a^{-1}(-H+b)\sigma^{-1}(g), & \text{if } 1, 2 \notin S \text{ or } 1, 2 \in S, \\ -a^{-1}(H^2 + H + b)\sigma^{-1}(g), & \text{if } 1 \in S, 2 \notin S, \\ a^{-1}\sigma^{-1}(g), & \text{if } 1 \notin S, 2 \in S; \end{cases} \end{aligned}$$

and

$$\mathcal{M}(\mathfrak{g}) = \{M(a, b, S) | a \in \mathbb{C}^*, b \in \mathbb{C}, S \subseteq \{1, 2\}\}.$$

Moreover $M(a, b, S)$ is simple if and only if $2b \notin \mathbb{Z}_+$ or $S \neq \emptyset, \{1, 2\}$.

So, to understand $\mathcal{M}(G)$, we only need to determine the actions of $E_i, F_i (1 \leq i \leq n)$ on each module in $\mathcal{M}(G_{\bar{0}})$. First, we have

Proposition 3.3. Let M_R be the module defined in Theorem 3.1. The following equations provides a G -module structure for the $G_{\bar{0}}$ -module M_R :

$$\begin{aligned} E_i \cdot g &= \begin{cases} \frac{\sqrt{2}}{2}\sigma_i(g), & i \in R, \\ \frac{\sqrt{2}}{2}(h_i - \frac{1}{2})\sigma_i(g), & i \notin R, \end{cases} ; \\ F_i \cdot g &= \begin{cases} \frac{\sqrt{2}}{2}(h_i + \frac{1}{2})\sigma_i^{-1}(g), & i \in R, \\ \frac{\sqrt{2}}{2}\sigma_i^{-1}(g), & i \notin R. \end{cases} \end{aligned} \quad (3.2)$$

where g is any polynomial in \mathcal{P} and i runs from 1 to n . We denote this G -module by M'_R . In particular, we have $M'_R \in \mathcal{M}(G)$ and M'_R is simple since it is a simple $G_{\bar{0}}$ -module.

Proof. It is clear that $M'_R \in \mathcal{M}(G)$ if it is a G -module. Also it is clear that M'_R is simple when $l \geq 2$.

If $l = 1$, then we have

$$\begin{aligned} E_{11} \cdot g &= 2E_1^2 \cdot g = \begin{cases} a_1^2 \sigma_1^2(g), & \text{if } 1 \in S, \\ a_1^2(h_1 - \frac{1}{2})(h_1 - \frac{3}{2})\sigma_1^2(g), & \text{if } 1 \notin S; \end{cases} \\ E_{\bar{1}\bar{1}} \cdot g &= -2F_1^2 \cdot g = \begin{cases} a_1^{-2}(h_1 + \frac{1}{2})(h_1 + \frac{3}{2})\sigma_1^{-2}(g), & \text{if } 1 \in S, \\ a_1^{-2}\sigma_1^{-2}(g), & \text{if } 1 \notin S. \end{cases} \end{aligned}$$

Hence

$$\text{Res}_{\mathcal{U}(G_0)}^{\mathcal{U}(G)} M'(a, S) \cong M(a_1^2, \{1\}, \frac{3}{8}) \text{ or } M(a_1^2, \{2\}, \frac{3}{8}),$$

which is a simple G_0 -module. Therefore if M'_R is a G -module, then it is a simple G -module.

To show that M'_R is a G -module, we need to check

$$\begin{aligned} E_{ij} \cdot g &= E_i \cdot E_j \cdot g + E_j \cdot E_i \cdot g, \\ E_{i\bar{j}} \cdot g &= E_i \cdot F_j \cdot g + F_j \cdot E_i \cdot g, \\ E_{\bar{i}\bar{j}} \cdot g &= F_i \cdot F_j \cdot g + F_j \cdot F_i \cdot g, \\ E_{ij} \cdot E_k \cdot g - E_k \cdot E_{ij} \cdot g &= 0, \\ E_{ij} \cdot F_k \cdot g - F_k \cdot E_{ij} \cdot g &= -\delta_{ik} E_j \cdot g - \delta_{jk} E_i \cdot g, \\ E_{\bar{i}\bar{j}} \cdot E_k \cdot g - E_k \cdot E_{\bar{i}\bar{j}} \cdot g &= \delta_{ik} F_j \cdot g + \delta_{jk} F_i \cdot g, \\ E_{\bar{i}\bar{j}} \cdot F_k \cdot g - F_k \cdot E_{\bar{i}\bar{j}} \cdot g &= 0, \\ E_{i\bar{j}} \cdot E_k \cdot g - E_k \cdot E_{i\bar{j}} \cdot g &= \delta_{jk} E_i \cdot g, \\ E_{i\bar{j}} \cdot F_k \cdot g - F_k \cdot E_{i\bar{j}} \cdot g &= -\delta_{ik} F_j \cdot g, \end{aligned}$$

where g is any polynomial in \mathcal{P} and i, j, k run from 1 to n .

We only verify M'_\emptyset is a G -module, the other cases can be verified similarly. The verification follows from direct computations. Indeed, from [16] and [6], we know that

$$\begin{aligned} E_{ij} \cdot g &= \left[h_i h_j - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j) + \frac{1}{2}(\frac{1}{2} + \delta_{ij}) \right] \sigma_i \sigma_j(g), \\ E_{\bar{i}\bar{j}} \cdot g &= \sigma_i^{-1} \sigma_j^{-1}(g), \\ E_{i\bar{j}} \cdot g &= (h_i - \frac{1}{2} + \frac{1}{2} \delta_{ij}) \sigma_i \sigma_j^{-1}(g). \end{aligned}$$

Thus, for any $g \in \mathcal{P}$,

$$E_i \cdot E_j \cdot g + E_j \cdot E_i \cdot g$$

$$\begin{aligned}
&= \frac{1}{2} \left[(h_i - \frac{1}{2})(h_j - \frac{1}{2} - \delta_{ij}) + (h_j - \frac{1}{2})(h_i - \frac{1}{2} - \delta_{ij}) \right] \sigma_i \sigma_j(g) \\
&= \frac{1}{2} \left[2h_i h_j - (1 + \delta_{ij})(h_i + h_j) + \frac{1}{2} + \delta_{ij} \right] \sigma_i \sigma_j(g) \\
&= \left[h_i h_j - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j) + \frac{1}{2}(\frac{1}{2} + \delta_{ij}) \right] \sigma_i \sigma_j(g) \\
&= E_{ij} \cdot g, \\
&E_i \cdot F_j \cdot g + F_j \cdot E_i \cdot g \\
&= \frac{1}{2} \left[(h_i - \frac{1}{2}) + (h_i - \frac{1}{2} + \delta_{ij}) \right] \sigma_i \sigma_j^{-1}(g) \\
&= (h_i - \frac{1}{2} + \frac{1}{2} \delta_{ij}) \sigma_i \sigma_j^{-1}(g) \\
&= E_{i\bar{j}} \cdot g, \\
&F_i \cdot F_j \cdot g + F_j \cdot F_i \cdot g \\
&= \frac{1}{2} (\sigma_i^{-1} \sigma_j^{-1}(g) + \sigma_i^{-1} \sigma_j^{-1}(g)) \\
&= \sigma_i^{-1} \sigma_j^{-1}(g) \\
&= E_{i\bar{j}} \cdot g.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&E_{ij} \cdot E_k \cdot g - E_k \cdot E_{ij} \cdot g \\
&= \frac{\sqrt{2}}{2} \left[\left(h_i h_j - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j) + \frac{1}{2}(\frac{1}{2} + \delta_{ij}) \right) (h_k - \frac{1}{2} - \delta_{ik} - \delta_{jk}) \right. \\
&\quad \left. - (h_k - \frac{1}{2}) \left((h_i - \delta_{ik})(h_j - \delta_{jk}) - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j - \delta_{ik} - \delta_{jk}) \right) \right. \\
&\quad \left. + \frac{1}{2}(\frac{1}{2} + \delta_{ij}) \right] \sigma_i \sigma_j \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \left[\frac{1}{2} \delta_{ij} \delta_{ik} h_j + \frac{1}{2} \delta_{ij} \delta_{jk} h_i - \delta_{ik} \delta_{jk} h_k + \frac{1}{2} \delta_{ik} \delta_{jk} \right. \\
&\quad \left. - \frac{1}{4} \delta_{ij} (\delta_{ik} + \delta_{jk}) \right] \sigma_i \sigma_j \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \delta_{ik} \delta_{jk} (\frac{1}{2} h_k + \frac{1}{2} h_k - h_k + \frac{1}{2} - \frac{1}{4}(1 + 1)) \sigma_i \sigma_j \sigma_k(g) \\
&= 0, \\
&E_{ij} \cdot F_k \cdot g - F_k \cdot E_{ij} \cdot g
\end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2} \left[\left(h_i h_j - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j) + \frac{1}{2}\left(\frac{1}{2} + \delta_{ij}\right) \right) \right. \\
&\quad \left. - \left((h_i + \delta_{ik})(h_j + \delta_{jk}) - \frac{1}{2}(1 + \delta_{ij})(h_i + h_j + \delta_{ik} + \delta_{jk}) \right) \right. \\
&\quad \left. + \frac{1}{2}\left(\frac{1}{2} + \delta_{ij}\right) \right) \sigma_i \sigma_j \sigma_k^{-1}(g) \\
&= \frac{\sqrt{2}}{2} \left[-\delta_{ik} h_j - \delta_{jk} h_i - \delta_{ik} \delta_{jk} + \frac{1}{2}(\delta_{ik} + \delta_{jk}) + \frac{1}{2} \delta_{ij}(\delta_{ik} + \delta_{jk}) \right] \sigma_i \sigma_j \sigma_k^{-1}(g) \\
&= -\frac{\sqrt{2}}{2} \delta_{ik} \left(h_j - \frac{1}{2} \right) \sigma_j(g) - \frac{\sqrt{2}}{2} \delta_{jk} \left(h_i - \frac{1}{2} \right) \sigma_i(g) \\
&= -\delta_{ik} E_j \cdot g - \delta_{jk} E_i \cdot g.
\end{aligned}$$

Since

$$E_{i\bar{j}} \cdot F_k \cdot g - F_k \cdot E_{i\bar{j}} \cdot g = 0$$

is clear, we only need to verify the remaining three relations. The computations are

$$\begin{aligned}
&E_{i\bar{j}} \cdot E_k \cdot g - E_k \cdot E_{i\bar{j}} \cdot g \\
&= \frac{\sqrt{2}}{2} \left[\left(h_k - \frac{1}{2} + \delta_{ik} + \delta_{jk} \right) - \left(h_k - \frac{1}{2} \right) \right] \sigma_i^{-1} \sigma_j^{-1} \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} (\delta_{ik} + \delta_{jk}) \sigma_i^{-1} \sigma_j^{-1} \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \delta_{ik} \sigma_j^{-1}(g) + \frac{\sqrt{2}}{2} \delta_{jk} \sigma_i^{-1}(g) \\
&= \delta_{ik} F_j \cdot g + \delta_{jk} F_i \cdot g, \\
&E_{i\bar{j}} \cdot F_k \cdot g - F_k \cdot E_{i\bar{j}} \cdot g \\
&= \frac{\sqrt{2}}{2} \left[\left(h_i - \frac{1}{2} + \frac{1}{2} \delta_{ij} \right) - \left(h_i - \frac{1}{2} + \frac{1}{2} \delta_{ij} + \delta_{ik} \right) \right] \sigma_i \sigma_j^{-1} \sigma_k^{-1}(g) \\
&= -\frac{\sqrt{2}}{2} \delta_{ik} \sigma_i \sigma_j^{-1} \sigma_k^{-1}(g) \\
&= -\frac{\sqrt{2}}{2} \delta_{ik} \sigma_j^{-1}(g) \\
&= -\delta_{ik} F_j \cdot g.
\end{aligned}$$

and

$$E_{i\bar{j}} \cdot E_k \cdot g - E_k \cdot E_{i\bar{j}} \cdot g$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2} \left[(h_i - \frac{1}{2} + \frac{1}{2}\delta_{ij})(h_k - \frac{1}{2} - \delta_{ik} + \delta_{jk}) \right. \\
&\quad \left. - (h_k - \frac{1}{2})(h_i - \frac{1}{2} + \frac{1}{2}\delta_{ij} - \delta_{ik}) \right] \sigma_i \sigma_j^{-1} \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \left[(h_i - \frac{1}{2} + \frac{1}{2}\delta_{ij})(\delta_{jk} - \delta_{ik}) + \delta_{ik}(h_i - \frac{1}{2}) \right] \sigma_i \sigma_j^{-1} \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \left[-\frac{1}{2}\delta_{ik}\delta_{ij} + \delta_{jk}(h_i - \frac{1}{2}) + \frac{1}{2}\delta_{ij}\delta_{jk} \right] \sigma_i \sigma_j^{-1} \sigma_k(g) \\
&= \frac{\sqrt{2}}{2} \delta_{jk}(h_i - \frac{1}{2}) \sigma_i(g) \\
&= \delta_{jk} E_i \cdot g.
\end{aligned}$$

□

Since $\{E_i, F_i, h_i | 1 \leq i \leq n\}$ is a set of generators for G satisfying relations (2.1), we know that M'_R is uniquely determined by equation (3.2) and $h_i \cdot g = h_i g$. Moreover, every module M in $\mathcal{M}(G)$ is uniquely determined by the actions of $E_i, F_i (1 \leq i \leq n)$ on M .

Theorem 3.4. *Let G be the basic Lie superalgebra $\mathfrak{osp}(1|2n)$. Then*

$$\mathcal{M}(G) = \{F_a(M'_R) | a \in (\mathbb{C}^*)^n, R \subseteq \{1, 2, \dots, n\}\}.$$

Moreover, since F_a is an auto-equivalence, any module in $\mathcal{M}(G)$ is simple.

Proof. Let $M \in \mathcal{M}(G)$, then $M_1 = \text{Res}_{G_0}^G M \in \mathcal{M}(G_0)$. Following from (2.1), we have

$$p_i \sigma_i(q_i) + q_i \sigma_i^{-1}(p_i) = h_i, i = 1, 2, \dots, n,$$

where $p_i = E_i \cdot 1, q_i = F_i \cdot 1$. Thus,

$$q_i \sigma_i^{-1}(p_i) = \frac{1}{2}(h_i + \frac{1}{2}).$$

Therefore, (p_i, q_i) is $(a_i, \frac{a_i^{-1}}{2}(h_i + \frac{1}{2}))$ or $(\frac{a_i}{2}(h_i - \frac{1}{2}), a_i^{-1})$ for some $a = (a_1, \dots, a_n) \in (\mathbb{C}^*)^n$.

If $M_1 \cong F_c(M_R)$ for some $c = (c_1, \dots, c_n) \in (\mathbb{C}^*)^n$, then

$$\{E_i, F_{i+1}\} = E_{i, \overline{i+1}}, i = 1, \dots, n-1,$$

and

$$\{E_n, E_n\} = E_{nn}$$

imply that

$$c_n = a_n^2, c_i a_{i+1} = a_i, i = 1, \dots, n-1$$

and

$$(p_i, q_i) = \begin{cases} (\frac{\sqrt{2}a_i}{2}(h_i - \frac{1}{2}), \frac{\sqrt{2}a_i^{-1}}{2}), & \text{if } i \in R, \\ (\frac{\sqrt{2}a_i}{2}, \frac{\sqrt{2}a_i^{-1}}{2}(h_i + \frac{1}{2})), & \text{if } i \notin R. \end{cases}$$

Hence,

$$M \cong F_{\pm a}(M'_R)$$

with $a_i = \sqrt{c_n} \prod_{j=i}^{n-1} c_j$. □

Acknowledgements. This work was done during the first author's visit at Wilfrid Laurier University. The hospitality of Wilfrid Laurier University is gratefully acknowledged. The first author also thanks the China Scholar Council for the financial support. This work was supported by National Natural Science Foundation of China (Grant 11401551).

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(Received March 14, 2015)